NYU | PARIS

MATH-UA 9252 · Numerical Analysis

Midterm exam

Fall $(11111101010)_2$ · Urbain Vaes

Name: _____

Pledge of academic integrity

I pledge that I will complete this exam on my own, without giving or receiving help from anyone, and without using any notes, devices or other materials not explicitly allowed.

Signature: _____

Exercise	Topic	Marks	Score
Question 1	Floating point arithmetic	15	
Implementation exercise 1	Julia basics	10	
Question 2	Interpolation	10	
Question 3	Interpolation (open-ended)	5	
Implementation exercise 2	Approximation	5	
Question 4	Numerical integration	10	
Question 5	Numerical integration (open-ended)	5	
Implementation exercise 3	Numerical integration	10	
Question 6 (bonus)	Numerical integration (open-ended)	+5	
	Total	**70**	

Question 1 Floating point arithmetic — 15 marks

True or false? Answer in the grid below.

Answer grid	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	B1	B2
True	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
False	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

1. Let $(\bullet)_2$ denote binary representation. Then $(0.0111)_2 + (0.1001)_2 = (1.0)_2$.
2. Let ε_{32} denote the machine epsilon for the `Float32` format, i.e. `eps(Float32)`. Then the number 4 is representable exactly in this format, and the next representable number is $4 + 4\varepsilon_{32}$.
3. It holds that $(100)_2 \times (0.0110)_2 = (11)_2$.
4. In Julia, `Float64(.75) == Float32(.75)` evaluates to `true`.
5. In Julia, the expression `0.1 + 0.2 == 0.3` evaluates to `true`.
6. It holds that $\left(0.\overline{011}\right)_2 = \frac{3}{7}$.
7. Machine multiplication $\hat{*}$ is a *commutative* operation: for any two numbers $x, y \in \mathbf{F}_{64}$, it holds that $x \,\hat{*}\, y = y \,\hat{*}\, x$.
8. There are exactly as many `Float64` numbers in the interval $[1, 2)$ as in the interval $[2, 4)$.
9. Let ε denote the machine epsilon of a binary floating point format. Then the spacing between successive floating point numbers in the interval $\left[\frac{1}{2}, 1\right)$ is equal to 2ε.
10. In Julia, the expression `1 + eps()/4 == 1` evaluates to `true`.
11. It holds that $(101.101)_2 = 5.625$.
12. A real number has a finite representation in base 2 if and only if it is a rational number.
13. The machine epsilon of a floating point format is the smallest strictly positive number that can be represented exactly in the format.
14. If `x` is a `Float16` number and `y` is a `Float64` number, then the result of `x + y` is a `Float64` number.
15. In Julia, the expression `eps()/3 + eps()/3 + 1 > 1` evaluates to `true`.

B1. **Bonus.** In Julia, the expression `sqrt(1 + 4eps()) == 1 + 2eps()` evaluates to `true`.

Explain briefly

B2. **Bonus.** In Julia, the expression `(1 + 2eps()/3) - 1 == 2eps()/3` evaluates to `true`.

Explain briefly

Implementation exercise 1 Julia basics — 10 marks

Let $f(x) = \mathrm{e}^{-x}\cos(\pi x)$. Write Julia code performing the following tasks, *without using any loop* (comprehensions are allowed). You may assume that the `Plots` library is loaded.

▶ **Functions.** Define a Julia function `f` that returns $f(x)$ for a number `x`.

▶ **Broadcasting.** Let `x = LinRange(0, 2, 101)`. Using the function `f`, construct the vector `y` containing the values of f at the points of `x`.

▶ **Slicing.** Using `x` and `y`, construct the vector of the 100 finite difference approximations $\frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ of the derivative f'.

▶ **Matrices.** Let `z = [0, 0.5, 1.5, 2]`. Construct the matrix $\mathsf{V} \in \mathbf{R}^{4\times 4}$ whose entry on the i-th row and j-th column is $V_{ij} = z_i^{j-1}$.

▶ **Plotting.** Plot on the same graph, over the interval $[0, 2]$, the function f together with the function $g(x) = \mathrm{e}^{-x}$, ideally with a legend indicating which curve is which.

Question 2 Interpolation — 10 marks

Are the following statements true or false? Justify your answers. Recall that $\mathcal{P}_d$ denotes the set of polynomials of degree at most d.

1. Assume that $x_0 < x_1 < x_2$ and y_0, y_1, y_2 are given real numbers. Then there exists a *unique* polynomial $p \in \mathcal{P}_3$ such that $p(x_i) = y_i$ for all $i \in \{0, 1, 2\}$. Prove or disprove from first principles, i.e. without merely invoking a theorem from class.

Justification

2. Assume that $x_0 < x_1 < \cdots < x_n$, and let $\ell_0, ..., \ell_n \in \mathcal{P}_n$ denote the associated Lagrange basis polynomials, so that $\ell_i(x_j) = 1$ if $i = j$ and $\ell_i(x_j) = 0$ otherwise. Then it holds that

$$\forall x \in \mathbf{R}, \quad \sum_{i=0}^{n} \ell_i(x) = 1.$$

Justification

3. Let $p \in \mathcal{P}_d$ be a polynomial of degree $d \geq 2$, and let $q : \mathbf{R} \to \mathbf{R}$ be given by

$$q(x) = p(x + 2) - 2p(x + 1) + p(x).$$

Then it holds that $q \in \mathcal{P}_{d-2}$.

Hint: it may be useful to express q using the difference operator $\Delta f(x) := f(x + 1) - f(x)$.

Justification

4. Let $f \in C^\infty[a, b]$, and let f_n denote the *piecewise linear* interpolant of f on a uniform mesh of $[a, b]$ with n cells, of size $h_n = \frac{b-a}{n}$. Then it holds that

$$\lim_{n \to \infty} n \left(\max_{x \in [a,b]} |f(x) - f_n(x)| \right) = 0.$$

Justification

Question 3 Interpolation (open-ended) — 5 marks

A continuous function $f : [a, b] \to \mathbf{R}$ can be approximated either by a single polynomial of degree n interpolating f at $n + 1$ nodes, or by a piecewise polynomial of fixed low degree (for example, piecewise linear) interpolating f on a mesh of $[a, b]$ with n cells. Compare these two approaches. For example, how does the interpolation error behave as n increases in each case, and on which properties of f and of the nodes does this behavior depend? In which situations would you prefer one approach over the other?

Illustrate your answer with examples of your choice, but go beyond merely stating formulas.

Implementation exercise 2 Approximation

5 marks

Write Julia code that computes and plots the polynomial $p \in \mathcal{P}_2$ that best approximates the following data in the least squares sense, i.e. that minimizes

$$\sum_i |p(x_i) - y_i|^2,$$

for the following data:

x_i	0	1	2	3	4	5
y_i	1.1	1.9	5.2	9.8	17.1	26.0

The plot should display both the data points and the graph of the polynomial over an appropriate range. Do not use any other library than the ones already imported.

```
using LinearAlgebra
using Plots
# Write your code here
```

Question 4 Numerical integration

10 marks

Are the following statements true or false? Justify your answers.

1. The degree of precision of the following quadrature rule is 2:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \frac{1}{3}u(-1) + \frac{4}{3}u(0) + \frac{1}{3}u(1).$$

Justification

2. The degree of precision of the following quadrature rule is 5:

$$\int_{0}^{1} u(x)\,\mathrm{d}x \approx u\left(\frac{1}{2}\right) + \frac{1}{24}u''\left(\frac{1}{2}\right).$$

Justification

3. Let $x_1 < \cdots < x_n$ be distinct nodes in $[-1, 1]$. If the quadrature rule

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i u(x_i)$$

has a degree of precision at least $n-1$, then its weights are given by

$$w_i = \int_{-1}^{1} \ell_i(x)\,\mathrm{d}x, \qquad \ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \qquad i = 1, \ldots, n.$$

Justification

4. Let $x_1, ..., x_n$ and $w_1, ..., w_n$ denote the nodes and weights of the n-point Gauss–Legendre rule on $[-1, 1]$. Then the n-point Gauss–Legendre rule on a general interval $[a, b]$ is given by

$$\int_a^b u(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i u\left(a + \frac{b-a}{2}(x_i + 1)\right).$$

Justification

5. There exist nodes $x_1, x_2 \in [-1, 1]$ and weights $w_1, w_2 \in \mathbf{R}$ such that the following quadrature rule has a degree of precision equal to 4:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx w_1 u(x_1) + w_2 u(x_2).$$

Justification

Question 5 Numerical integration (open-ended) — 5 marks

Composite quadrature rules are obtained by partitioning the integration interval into N cells of size h and applying a simple quadrature rule within each cell. Discuss how the error of a composite rule behaves as $h \to 0$, and how this behavior is related to the degree of precision of the underlying simple rule.

Illustrate your answer with Simpson's rule, considering both the simple rule and its composite version, but go beyond merely stating formulas.

Implementation exercise 3 Numerical integration — 10 marks

Consider the following quadrature rule:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \frac{1}{12}\left(11u\left(-\frac{3}{5}\right) + u\left(-\frac{1}{5}\right) + u\left(\frac{1}{5}\right) + 11u\left(\frac{3}{5}\right)\right).$$

▶ (**2 marks**) Write a function `simple_rule(u)` that returns the approximation of $\int_{-1}^{1} u(x)\,\mathrm{d}x$ given by this quadrature rule.

```
function simple_rule(u)
    # Write your code here



end
```

▶ (**3 marks**) Use the function `simple_rule` to determine numerically the degree of precision of the rule, by applying it to the monomials $u(x) = x^d$ for $d \in \{0, 1, 2, ...\}$. What is the degree of precision?

```
# Write your code here
```

- (**5 marks**) Write a function `composite_rule(u, a, b, N)` that returns an approximation of the integral

$$\int_a^b u(x)\,\mathrm{d}x,$$

obtained by partitioning the integration interval $[a, b]$ into N cells, and applying the quadrature rule within each cell. Your function should call the function `simple_rule` you wrote in the previous item.

```
function composite_rule(u, a, b, N)
    # Write your code here



end
```

Question 6 (bonus) Numerical integration (open-ended) — 5 bonus marks

Let $\hat{I}(h)$ denote the approximation of an integral I given by a composite quadrature rule with step size h, and assume that

$$\hat{I}(h) = I + Ch^p + \mathcal{O}(h^q),$$

for some constant $C \neq 0$ independent of h and some exponents $q > p > 0$. Explain how Richardson extrapolation can be used to obtain a more accurate approximation from the two approximations $\hat{I}(h)$ and $\hat{I}(2h)$.

Illustrate your answer with the composite trapezoidal rule, but go beyond merely stating formulas.