True or false? (unless otherwise specified)

1. The only polynomial of any degree such that $p(-1) = 1$, $p(0) = 0$ and $p(1) = 1$ is the quadratic polynomial $p(x) = x^2$.

2. Let $p \in \mathcal{P}_3$ be given by $p(x) = x^3 + 3x^2 - x + 1$ and let $q \in \mathcal{P}_2$ denote the polynomial interpolation of p at $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. Then there exists a constant $C \neq 0$ such that error satisfies

$$\forall x \in \mathbb{R}, \qquad p(x) - q(x) = C(x - x_0)(x - x_1)(x - x_2)$$

3. Given $x_0 < x_1 < \ldots < x_n \in \mathbb{R}$ and $y_0, y_1, \ldots, y_n \in \mathbb{R}$, there exist infinitely many polynomials $p \in \mathcal{P}_{n+1}$ such that $p(x_i) = y_i$ for all $i \in \{0, 1, \ldots, n\}$.

4. In interpolation, the choice of interpolating points can have an influence on the interpolation error.

5. Suppose that $f \colon [-1, 1] \to \mathbb{R}$ is given by $f(x) = e^x$, and for any $n \in \mathbb{N}$, let $f_n \in \mathcal{P}_n$ denote the polynomial interpolating f at $n + 1$ equidistant points $-1 = x_0 < x_1 < \ldots < x_n = 1$. Then

$$\lim_{n \to \infty} \left(\max_{-1 \leqslant x \leqslant 1} \bigl| f(x) - f_n(x) \bigr| \right) = 0.$$

6. There exists a polynomial p such that

$$\forall n \in \mathbb{N}, \qquad p(n) = 2^n \,.$$

   Hint: Let $s(n) = 2^n$. If s were a polynomial, there would exist $\alpha \in \mathbb{N}$ such that $\Delta^\alpha s = 0$.

7. In Julia, if `A` is a matrix, then `A[:, 1:2]` gives the submatrix containing the first two rows of `A`.

8. In Julia, if `A` is a matrix, then `A[1:end .!= 2, 1:end .!= 2]` gives the matrix obtained by removing the second column and the second row.

9. In Julia, typing `]` in a REPL (command line) enables to access package mode, from which new packages can be installed.

10. In the following code, `p` is the interpolating polynomial through the data in `x` and `y`.

```julia
using Plots
x = [0, 1, 2, 3]
y = [1, 2, 1, 2]

function p(x)
    return (y[1]
            + diff(y)[1] * x
            + 1/2 * diff(diff(y))[1] * x * (x-1)
            + 1/6 * diff(diff(diff(y)))[1] * x * (x-1) * (x-2))
end

plot(p, xlims=(0, 5))
scatter!(x, y)
```

11. **Bonus**: Obtain an explicit expression for $S(N) := \sum_{n=0}^{N} n^3$ by Gregory–Newton interpolation.