1. Suppose that $x_0 < x_1 < \cdots < x_n$ and $y_0, y_1, \ldots, y_n$ are given real numbers. Then there exists a unique polynomial $p \in \mathcal{P}_n$ such that $p(x_i) = y_i$ for all $i \in \{0, \ldots, n\}$.

2. Suppose that $f\colon [-1,1] \to \mathbb{R}$ is given by $f(x) = \mathrm{e}^{-2x}$, and for any $n \in \mathbb{N}$, let $f_n \in \mathcal{P}_n$ denote the polynomial interpolating f at $n+1$ equidistant points $-1 = x_0 < x_1 < \ldots < x_n = 1$. Then
$$\lim_{n\to\infty} \left(\max_{-1 \leqslant x \leqslant 1} \bigl| f(x) - f_n(x) \bigr| \right) = 0.$$

3. Define $\mathcal{F}\colon \mathcal{P}_n \to \mathbf{R}$ as the function which, to a polynomial $p \in \mathcal{P}_n$, associates its maximum absolute value over the interval $[-1,1]$, that is
$$\mathcal{F}(p) = \max_{x \in [-1,1]} |p(x)| \ .$$
Then it holds that $\mathcal{F}(p) \geqslant \mathcal{F}(T_n)$ for all $p \in \mathcal{P}_n$, where T_n is the Chebyshev polynomial of degree n.

4. Suppose that $f\colon [-1,1] \to \mathbb{R}$ is the Runge function $f(x) = \frac{1}{1+25x^2}$. For any $n \in \mathbb{N}$, let $f_n \in \mathcal{P}_n$ denote the polynomial interpolating f at $n+1$ *equidistant* points, and let $g_n \in \mathcal{P}_n$ denote the polynomial interpolating f at the $n+1$ *Chebyshev* points (the roots of T_{n+1}). Then it holds that
$$\lim_{n\to\infty} \left(\max_{-1 \leqslant x \leqslant 1} \bigl| f(x) - f_n(x) \bigr| \right) = +\infty \,, \qquad \lim_{n\to\infty} \left(\max_{-1 \leqslant x \leqslant 1} \bigl| f(x) - g_n(x) \bigr| \right) = 0 \,.$$

5. Suppose that $x_0 < x_1 < \cdots < x_n$ and $y_0, y_1, \ldots, y_n$ are given real numbers. Then for any $m \in \{0, 1, \ldots\}$, there exists exactly one polynomial $p \in \mathcal{P}_m$ such that
$$\frac{1}{n} \sum_{i=0}^{n} \Bigl| p(x_i) - y_i \Bigr|^2 = 0.$$

6. Given $x_0 < x_1 < \ldots < x_n \in \mathbb{R}$ and $y_0, y_1, \ldots, y_n \in \mathbb{R}$, and let $m \geqslant n$. There exists a unique $p \in \mathcal{P}_m$ such that the following quantity is minimized:
$$\frac{1}{n} \sum_{i=0}^{n} \Bigl| p(x_i) - y_i \Bigr|^2.$$

7. Given $x_0 < x_1 < \ldots < x_n \in \mathbb{R}$ and $y_0, y_1, \ldots, y_n \in \mathbb{R}$ with $n = 10$, there exists a unique polynomial $p(x) = ax + b$ that minimizes
$$J(a,b) = \frac{1}{2} \sum_{i=0}^{n} \Bigl| y_i - p(x_i) \Bigr|^2.$$

8. In Julia, if `v` is a vector of size 5, then `v[[1, 2, 5]]` returns a vector with the first, second and fifth element of `v`.

9. In Julia, if `v` is a vector of size 5, then `v[[true, true, false, true, false]]` returns a vector with the first, second and fourth element of `v`.

10. In Julia, if `A` is a 10×10 matrix, then `A[mod.(1:end, 2) .== 0, mod.(1:end, 2) .== 0]` gives the matrix obtained by removing the second column and the second row.

Answer key

1. **True.** This is the fundamental theorem of interpolation: the set of nodes $x_0 < x_1 < \cdots < x_n$ leads to a nonsingular Vandermonde system, so there is a unique $p \in \mathcal{P}_n$ solving $p(x_i) = y_i$ for all i.

2. **True.** The function $x \mapsto \mathrm{e}^{-2x}$ is entire, so polynomial interpolation on equidistant nodes converges uniformly on $[-1, 1]$ as $n \to \infty$.

3. **False.** The minimal property goes in the *opposite* direction: among monic polynomials of degree n, T_n has the smallest possible sup-norm on $[-1, 1]$. For example, the zero polynomial satisfies $\mathcal{F}(0) = 0 < 1 = \mathcal{F}(T_n)$.

4. **True.** This is the statement of the Runge phenomenon: interpolation at equidistant points diverges for the Runge function, while interpolation at the Chebyshev nodes (the roots of T_{n+1}) converges uniformly.

5. **False.** For $m < n$ interpolation with a degree-m polynomial is in general impossible (the condition imposes $n + 1$ equalities), and for $m > n$ the polynomial is not unique.

6. **False.** When $m > n$ there are more coefficients than data points, so the design matrix does not have full column rank and the least-squares problem has infinitely many minimizers (indeed the minimum can be zero, attained by the interpolating polynomial).

7. **True.** With 11 data points and only 2 unknown parameters a, b, the normal equations are a 2×2 system whose matrix is nonsingular as long as the nodes are not all identical, so the minimizer is unique.

8. **True.** Indexing a vector with a vector of indices selects the corresponding entries in that order: here the first, second and fifth elements of `v`.

9. **True.** Indexing with a Boolean mask of the same length keeps exactly the entries in the positions where the mask is `true`: the first, second and fourth elements.

10. **False.** `mod.(1:end, 2) .== 0` produces the mask `[false, true, false, true, ...]`, which selects the even-indexed rows and the even-indexed columns, giving the 5×5 submatrix of rows and columns $2, 4, 6, 8, 10$ — not the matrix obtained by deleting row 2 and column 2.