1. Given real numbers $x_0 < x_1 < x_2 < x_3$ and y_0, y_1, y_2, y_3, the unique $p \in \mathcal{P}_3$ such that $p(x_i) = y_i$ for all $i \in \{0, \dots, n\}$ is given by

$$p(x) = y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

2. Gregory–Newton interpolation is well suited for incremental interpolation.

3. Suppose that $f\colon [-1, 1] \to \mathbf{R}$ is given by $f(x) = (1+25x^2)^{-1}$. For integers $n, m \geq 1$, let $f_{n,m} : [-1, 1] \to \mathbb{R}$ denote the piecewise interpolant obtained by dividing $[-1, 1]$ into n equal subintervals and performing an equidistant polynomial interpolation of degree m on each subinterval. The resulting function $f_{n,m}$ is thus a piecewise polynomial approximation of f. Then for any fixed $m \geqslant 0$, it holds that

$$\lim_{n \to \infty} \left(\max_{-1 \leqslant x \leqslant 1} \bigl| f(x) - f_{n,m}(x) \bigr| \right) = +\infty \,.$$

4. Consider the function $F_n\colon \mathbf{R}^n \to \mathbf{R}$ given by

$$F_n(x_1, \dots, x_n) = \max_{x \in [-1,1]} \Bigl| (x - x_1) \dots (x - x_n) \Bigr|$$

Then for all $n \in \{1, 2, \dots\}$, the minimum of F_n over $\mathbf{R}^n$ is given by $2/n$.

5. Given (not necessarily distinct) real numbers $x_0, \dots, x_n$ and $y_0, \dots, y_n$, there exists a unique polynomial of degree n such that

$$\forall i \in \{0, \dots, n\}, \qquad p(x_i) = y_i.$$

6. Using any method you like, write in canonical form the polynomial p interpolating the points $(0, 1)$, $(1, 4)$, $(2, 15)$, $(3, 40)$.

   Answer:

7. In Julia, if `v` is a vector of size 5, then `v[v .> 0]` returns a vector with all the positive elements of `v`.

8. In Julia, if `v` is a vector of size 3, then `v[[1, 1, 1]]` returns the whole vector `v`.

9. In Julia, if `A` is a 10×10 matrix, then `A[mod.(1:end, 3) .== 2, :]` gives the matrix obtained by keeping only the third and sixth rows.

10. Write the expression of the polynomial p defined in the following Julia code

```
x = [1, 2, 3, 4]
p(z) = prod(z .- x[[1, 3, 4]]) / prod(x[2] .- x[[1, 3, 4]])
```

   Answer:

11. **(Bonus)** Given real numbers $x_0 < \dots < x_n$ and $y_0, \dots, y_n$, find the *constant* polynomial $p(x) = c$ that best approximates the data in the least–squares sense. Equivalently, find $c \in \mathbf{R}$ that minimizes

$$\sum_{i=0}^{n} |y_i - c|^2 \,.$$