Throughout this quiz, we consider the following linear system for large n:

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \mathsf{A} \in \mathbf{R}^{n\times n}, \qquad \boldsymbol{b} \in \mathbf{R}^n. \tag{1}$$

We assume throughout that **matrix A is symmetric and positive definite,** and we denote by $\boldsymbol{x}_*$ the exact solution to the linear system (1). Are the following statements true or false?

1. The matrix A is invertible.
2. The matrix A is diagonalizable.
3. The eigenvalues of A are all strictly positive.
4. The spectral radius $\rho(\mathsf{A})$ is strictly positive.
5. There exists a unique lower triangular matrix $\mathsf{C} \in \mathbf{R}^{n\times n}$ such that $\mathsf{A} = \mathsf{C}\mathsf{C}^T$.
6. The condition number $\kappa_2(\mathsf{A})$ is less than or equal to the spectral radius $\rho(A)$.
7. Richardson's iterative method is convergent for $\omega = 0.01$.
8. For a splitting $\mathsf{A} = \mathsf{M} - \mathsf{N}$ with M an invertible matrix, consider the iterative method

   $$\mathsf{M}\boldsymbol{x}^{(k+1)} = \mathsf{N}\boldsymbol{x}^{(k)} + \boldsymbol{b}. \tag{2}$$

   Richardson's iteration is a particular case of this iterative method.
9. If $\rho(\mathsf{M}^{-1}\mathsf{N}) < 1$, then the basic iterative method (3) is convergent.
10. Let $\mathsf{M}_{\rm GS} - \mathsf{N}_{\rm GS}$ denote the Gauss–Seidel splitting. The basic iterative method (3) is convergent for this splitting.
11. If $\mathsf{M} = \mathsf{A}$ and $\mathsf{N} = 0$ in (3), then the iterative method converges in just one iteration, which however amounts to solving the initial problem (1).
12. Assume that A is symmetric and positive definite. Then $\boldsymbol{x}_*$ is a solution of (1) *if and only if* $\boldsymbol{x}_*$ is a minimizer of the following convex function:

    $$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}.$$

13. Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}_*$ and $\boldsymbol{r}^{(k)} = \mathsf{A}\boldsymbol{x}^{(k)} - \boldsymbol{b}$ denote the error and residual, respectively. Then

    $$\|\boldsymbol{e}^{(k)}\| \leq \|\mathsf{A}\|\,\|\boldsymbol{r}^{(k)}\|$$

14. If the residual $\boldsymbol{r}^{(k)}$ is zero, then so is the error $\boldsymbol{e}^{(k)}$.
15. The following iterative method is convergent:

    $$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \left(\frac{(\boldsymbol{r}^{(k)})^T\boldsymbol{r}^{(k)}}{(\boldsymbol{r}^{(k)})^T\mathsf{A}\boldsymbol{r}^{(k)}}\right)\boldsymbol{r}^{(k)}. \tag{3}$$

16. In Julia, `A[:, 3]` selects the third row of matrix `A`.
17. In Julia, `A[1:5, :]` returns the submatrix containing the first five rows of `A`.
18. In Julia, `A[1, :] + b` returns the sum of the first row of `A` and the vector `b`.
19. In Julia, `b[1:end .== 5]` returns the fifth element of `b`.
20. In Julia, `b[b .> 0]` returns a vector containing all and only the positive elements of `b`.