Throughout this quiz, we consider the following linear system:

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \mathsf{A} \in \mathbf{R}^{n\times n}, \qquad \boldsymbol{b} \in \mathbf{R}^n. \tag{1}$$

1. The computational cost of the backward substitution algorithm, used to solve (1) when A is an upper triangular matrix, scales as $\mathcal{O}(n^2)$.
2. For a general invertible matrix A, the number of floating point operations required to calculate its LU decomposition scales as $\mathcal{O}(n^2)$.
3. If A is symmetric, then there exists a lower triangular matrix $\mathsf{C} \in \mathbf{R}^{n\times n}$ such that $\mathsf{A} = \mathsf{C}\mathsf{C}^T$.
4. The condition number $\kappa_2(\mathsf{A})$ of a square invertible matrix A is always strictly larger than 1.
5. Suppose that A is a symmetric, positive definite matrix. Then calculating the Cholesky decomposition of A requires more floating point operations than calculating the LU decomposition of A.
6. The spectral radius of a square matrix A is zero if and only if A is zero.
7. If the condition number of a matrix A is very large ($\gg 1$), then it is possible that rounding errors arising from floating point arithmetic will have a large impact on the accuracy of the numerical solution to the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ (calculated by LU decomposition followed by forward and backward substitution, for example).
8. Suppose that $\mathsf{A} \in \mathbf{R}^{n\times n}$ is symmetric and positive definite, and let $\boldsymbol{b} \in \mathbf{R}^n$. Consider the following iterative method for solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$:

   $$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega\left(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)}\right). \tag{2}$$

   This iteration converges to the exact solution of the linear system for all $\omega \in \mathbf{R}$.
9. The convergence speed of the iteration (2), for the optimal value of ω, is independent of the condition number $\kappa_2(\mathsf{A})$.
10. If A is symmetric positive definite, there always exists $\omega \in \mathbf{R}$ such that the iteration (2) converges.
11. If A is a nonzero matrix, then its norm $\|\mathsf{A}\|_2$ is strictly positive.
12. Suppose that we want to solve the linear systems $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ and $\mathsf{A}\boldsymbol{x} = \boldsymbol{c}$ by a direct method. To this end, the LU decomposition of A can be calculated only once.
13. The Jacobi basic iterative method is convergent if the matrix A is strictly diagonally dominant.
14. Assume that A is symmetric and positive definite. Then $\boldsymbol{x}_*$ is a solution of (1) *if and only if* $\boldsymbol{x}_*$ is a minimizer of the following convex function:

    $$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}.$$
15. The Gauss–Seidel basic iterative method is convergent if the matrix A is strictly diagonally dominant.
16. Let $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}_*$, where $\boldsymbol{x}_*$ is the exact solution to the linear system (1), and $\boldsymbol{x}^{(k)}$ is obtained from the iterative method (2). Then successive errors are related by the following equation:

    $$\boldsymbol{e}^{(k+1)} = (\mathsf{I} - \omega\mathsf{A})\boldsymbol{e}^{(k)}.$$
17. Assuming that the matrix `A` and the vector `b` are already defined, as well as the parameter ω, write (on paper) Julia code implementing 100 iterations of (2).