True or false? (unless otherwise specified)

1. If $\mathsf{A} \in \mathbf{R}^{n \times n}$ is singular (non-invertible), then for any vector $\boldsymbol{b} \in \mathbf{R}^n$, there exist infinitely many solutions to the linear system
$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}\,.$$

2. Suppose that $(\boldsymbol{x}_i)_{i \in \mathbf{N}}$ is a sequence in $\mathbf{R}^n$ and that $\boldsymbol{x}_* \in \mathbf{R}^n$. Then we have the equivalence
$$\lim_{i \to \infty} \|\boldsymbol{x}_i - \boldsymbol{x}_*\|_\infty = 0 \qquad \Leftrightarrow \qquad \lim_{i \to \infty} \|\boldsymbol{x}_i - \boldsymbol{x}_*\|_1 = 0\,.$$

3. For a vector norm $\|\bullet\|$ on $\mathbf{R}^n$, the *subordinate* or *induced* matrix norm is defined by
$$\|\mathsf{M}\| := \max\bigl\{\|\mathsf{M}\boldsymbol{x}\| : \|\boldsymbol{x}\| \leqslant 1\bigr\}\,.$$
Then it holds that $\|\mathsf{AB}\| \leqslant \|\mathsf{A}\|\|\mathsf{B}\|$ for all $\mathsf{A}, \mathsf{B} \in \mathbf{R}^{n \times n}$.

4. Suppose that $\mathsf{A} \in \mathbf{R}^{n \times n}$ is an invertible matrix and consider a splitting $\mathsf{A} = \mathsf{M} - \mathsf{N}$, with M an invertible matrix. Suppose that $\boldsymbol{b} \in \mathbf{R}^n$ is given and $\boldsymbol{x}^{(0)} = \boldsymbol{0} \in \mathbf{R}^n$. Consider the following iterative method:
$$\mathsf{M}\boldsymbol{x}^{(k+1)} = \mathsf{N}\boldsymbol{x}^{(k)} + \boldsymbol{b}\,, \tag{1}$$
Denote by $\boldsymbol{x}_*$ the exact solution to the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, and recall that, as we proved in class, the error $\boldsymbol{e}^{(k)} := \boldsymbol{x}^{(k)} - \boldsymbol{x}_*$ satisfies the equation
$$\boldsymbol{e}^{(k)} = (\mathsf{M}^{-1}\mathsf{N})^k \boldsymbol{e}^{(0)}\,. \tag{2}$$
Then the error satisfies the inequality
$$\|\boldsymbol{e}^{(k)}\|_\infty \leqslant \|\mathsf{M}^{-1}\mathsf{N}\|_\infty^k \|\boldsymbol{e}^{(0)}\|_\infty\,.$$

5. For the iterative method (1), the approximation $\boldsymbol{x}^{(k)}$ converges as $k \to \infty$ to the exact solution $\boldsymbol{x}_*$ *if and only if* the following inequality is satisfied: $\|\mathsf{M}^{-1}\mathsf{N}\|_\infty < 1$.

6. The Jacobi method is an iterative method of the form (1) for the splitting $\mathsf{M} = \mathsf{D}$ and $\mathsf{N} = -\mathsf{L} - \mathsf{U}$, where matrix D is the diagonal part of A, and L, U are the strictly lower and upper triangular parts, respectively. Then, for a general matrix A, each iteration of this method requires $\mathcal{O}(n)$ floating point operations.

7. Assume that $\boldsymbol{b} \in \mathbf{R}^n$ and that $\mathsf{A} \in \mathbf{R}^{n \times n}$ is symmetric and positive definite. Then a vector $\boldsymbol{x}_*$ satisfies the equation $\mathsf{A}\boldsymbol{x}_* = \boldsymbol{b}$ *if and only if*
$$f(\boldsymbol{x}_*) = \min_{\boldsymbol{x} \in \mathbf{R}^n} f(\boldsymbol{x}), \qquad \text{where } f(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}\,. \tag{3}$$

8. Assume that $\boldsymbol{b} \in \mathbf{R}^n$ and that $\mathsf{A} \in \mathbf{R}^{n \times n}$ is symmetric and positive definite. Assume additionally that the vectors $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ are A-conjugate, meaning that $\boldsymbol{e}_i^T\mathsf{A}\boldsymbol{e}_j = 0$ if $i \neq j$, and denote by $\boldsymbol{x}_*$ the exact solution to the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Then it holds that
$$\boldsymbol{x}_* = \frac{\boldsymbol{e}_1^T\boldsymbol{b}}{\boldsymbol{e}_1^T\mathsf{A}\boldsymbol{e}_1}\boldsymbol{e}_1 + \cdots + \frac{\boldsymbol{e}_n^T\boldsymbol{b}}{\boldsymbol{e}_n^T\mathsf{A}\boldsymbol{e}_n}\boldsymbol{e}_n\,.$$

9. Suppose that $\mathsf{A} \in \mathbf{R}^{2 \times 2}$ is symmetric, with a positive eigenvalue and a negative eigenvalue. Then the function f defined in (3) does not have a minimizer, and furthermore
$$\inf_{\boldsymbol{x} \in \mathbf{R}^n} f(\boldsymbol{x}) = -\infty\,.$$

10. The following code implements an iterative method for solving $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. (Note that the matrix A here is symmetric and positive definite.) In this case the method does not converge; the `while` loop never terminates.

```
A = [4 1 0; 1 4 1; 0 1 4]
b = [1.; 2.; 3.]
x = [0.; 0.; 0.]
r = A*x - b
while sqrt(r'r) > 1e-12 * sqrt(b'b)
    r = A*x - b
    w = r'r / (r'*A*r)
    x -= w * r
end
```

11. **Bonus**: The following code implements a basic iterative method based on a splitting. What is the name of the method implemented? Also give the explicit expressions of M and N.

```
A = [4 1 0; 1 4 1; 0 1 4]
b = [1.; 2.; 3.]
x = [0.; 0.; 0.]
for k in 1:100
    x[1] = (b[1] - A[1, 2] * x[2] - A[1, 3] * x[3]) / A[1, 1]
    x[2] = (b[2] - A[2, 1] * x[1] - A[2, 3] * x[3]) / A[2, 2]
    x[3] = (b[3] - A[3, 1] * x[1] - A[3, 2] * x[1]) / A[3, 3]
end
```

Your answer:

12. **Bonus:** Prove the equation (2) satisfied by the error for the basic iterative method based on a splitting.

Your answer:

Solutions

1. **False.** A singular matrix A does not necessarily yield infinitely many solutions. The system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has solutions only when $\boldsymbol{b} \in \text{range}(\mathsf{A})$; otherwise it has no solution. Example: if $\mathsf{A} = 0$, then no solution exists unless $\boldsymbol{b} = \mathbf{0}$.

2. **True.** In $\mathbb{R}^n$ all norms are equivalent. In particular,

$$\|\boldsymbol{v}\|_\infty \leq \|\boldsymbol{v}\|_1 \leq n\|\boldsymbol{v}\|_\infty,$$

so $\|\boldsymbol{x}_i - \boldsymbol{x}_*\|_\infty \to 0$ iff $\|\boldsymbol{x}_i - \boldsymbol{x}_*\|_1 \to 0$.

3. **True.** For any $\boldsymbol{x}$ with $\|\boldsymbol{x}\| \leq 1$,

$$\|\mathsf{AB}\boldsymbol{x}\| = \|\mathsf{A}(\mathsf{B}\boldsymbol{x})\| \leq \|\mathsf{A}\|\,\|\mathsf{B}\boldsymbol{x}\| \leq \|\mathsf{A}\|\,\|\mathsf{B}\|,$$

and maximizing over $\|\boldsymbol{x}\| \leq 1$ gives $\|\mathsf{AB}\| \leq \|\mathsf{A}\|\,\|\mathsf{B}\|$.

4. **True.** Using the error formula $\boldsymbol{e}^{(k)} = (\mathsf{M}^{-1}\mathsf{N})^k\boldsymbol{e}^{(0)}$ and submultiplicativity,

$$\|\boldsymbol{e}^{(k)}\|_\infty \leq \|(\mathsf{M}^{-1}\mathsf{N})^k\|_\infty\,\|\boldsymbol{e}^{(0)}\|_\infty \leq \|\mathsf{M}^{-1}\mathsf{N}\|_\infty^k\|\boldsymbol{e}^{(0)}\|_\infty.$$

5. **False.** The condition $\|\mathsf{M}^{-1}\mathsf{N}\|_\infty < 1$ is *sufficient* for convergence but not *necessary.* The correct necessary and sufficient condition is $\rho(\mathsf{M}^{-1}\mathsf{N}) < 1$, where $\rho(\cdot)$ denotes the spectral radius.

6. **False.** For a general dense matrix A, one Jacobi iteration requires $\mathcal{O}(n^2)$ operations: each of the n components requires summing approximately n terms. The cost is $\mathcal{O}(n)$ only for banded or sparse matrices.

7. **True.** Since A is symmetric positive definite, $f(\boldsymbol{x})$ is strictly convex and

$$\nabla f(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x} - \boldsymbol{b}.$$

Thus f is minimized exactly at points satisfying $\mathsf{A}\boldsymbol{x}_* = \boldsymbol{b}$.

8. **True.** Since the $\boldsymbol{e}_i$ form an A-conjugate basis, write $\boldsymbol{x}_* = \sum_{i=1}^n \alpha_i\boldsymbol{e}_i$. Taking $\boldsymbol{e}_j^\top\mathsf{A}(\cdot)$ gives $\boldsymbol{e}_j^\top\boldsymbol{b} = \alpha_j\boldsymbol{e}_j^\top\mathsf{A}\boldsymbol{e}_j$, hence

$$\boldsymbol{x}_* = \sum_{i=1}^n \frac{\boldsymbol{e}_i^\top\boldsymbol{b}}{\boldsymbol{e}_i^\top\mathsf{A}\boldsymbol{e}_i}\boldsymbol{e}_i.$$

9. **True.** If A has a positive and a negative eigenvalue, then along the eigenvector $\boldsymbol{v}$ associated with the negative eigenvalue $\lambda < 0$,

$$f(t\boldsymbol{v}) = \tfrac{1}{2}\lambda t^2 - t\boldsymbol{v}^\top\boldsymbol{b} \to -\infty \quad \text{as } |t| \to \infty.$$

Thus f has no minimizer and $\inf f = -\infty$.

10. **False (the claim of nonconvergence is false).** The code implements the steepest–descent method with exact line search: for A symmetric positive definite, this method always converges. Thus the loop should terminate (modulo tolerance issues), not run forever.

Bonus 1. The method is **Gauss–Seidel**. With the splitting $\mathsf{A} = \mathsf{D} + \mathsf{L} + \mathsf{U}$, the Gauss–Seidel iteration uses

$$\mathsf{M} = \mathsf{D} + \mathsf{L}, \qquad \mathsf{N} = -\mathsf{U},$$

so that

$$(\mathsf{D} + \mathsf{L})\boldsymbol{x}^{(k+1)} = -\mathsf{U}\boldsymbol{x}^{(k)} + \boldsymbol{b}.$$

Bonus 2. Starting from

$$\mathsf{M}\boldsymbol{x}^{(k+1)} = \mathsf{N}\boldsymbol{x}^{(k)} + \boldsymbol{b},$$

and using the fact that $\mathsf{M}\boldsymbol{x}_* = \mathsf{N}\boldsymbol{x}_* + \boldsymbol{b}$ (since $\mathsf{A} = \mathsf{M} - \mathsf{N}$ and $\mathsf{A}\boldsymbol{x}_* = \boldsymbol{b}$), subtract the equations to obtain

$$\mathsf{M}(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}_*) = \mathsf{N}(\boldsymbol{x}^{(k)} - \boldsymbol{x}_*).$$

Hence

$$\boldsymbol{e}^{(k+1)} = \mathsf{M}^{-1}\mathsf{N}\,\boldsymbol{e}^{(k)}.$$

Iterating yields

$$\boldsymbol{e}^{(k)} = (\mathsf{M}^{-1}\mathsf{N})^k\boldsymbol{e}^{(0)}.$$