Throughout the quiz, we consider the following scalar, nonlinear equation:

$$f(x) = 0, \qquad f\colon \mathbf{R} \to \mathbf{R}, \qquad x \in \mathbf{R}. \tag{1}$$

1. Regardless of the specific form of f, there exists a unique solution to (1).
2. There may or may not exist a solution to (1), depending on the specific form of f.

   If you answer true, justify with examples.

3. Suppose there exists a solution to (1). Then the solution is unique.

   If you answer false, justify with an counterexample.

4. Suppose that f is continuous and that $f(0)f(1) < 0$. Then there exists a solution $x_* \in (0, 1)$ to (1).
5. Suppose that f is continuous, that $f(0)f(1) < 0$, and that the bisection method is employed with $a = 0$ and $b = 1$ in order to find a root of f. Then it takes fewer than 100 iterations to obtain a approximation of the solution with an error smaller that 10^{-10}.
6. Suppose that we employ the chord method in order to find a solution to (1):

   $$x_{k+1} = x_k - \frac{f(x_k)}{\alpha}. \tag{2}$$

   Then this method converges for any value of the parameter $\alpha \in \mathbf{R} \setminus \{0\}$.

7. Suppose that $f(x) = x$ and that $\alpha = 2$. Then the chord method (2) converges.
8. The chord method may be rewritten as a fixed point iteration of the form

   $$x_{k+1} = F_{\text{chord}}(x_k)$$

   for an appropriate function F_{chord}. Write the expression of the function F_{chord}:

   $$F_{\text{chord}}(x) =$$

9. **(2 marks)** Consider a general fixed point iteration of the form

   $$x_{k+1} = F(x_k).$$

   Suppose that $F(x_*) = 0$ and that F is Lipschitz continuous with constant $L < 1$:

   $$\forall x, y \in \mathbf{R}, \qquad \bigl|F(x) - F(y)\bigr| \leq L|x - y|.$$

   Prove that

   $$|x_k - x_*| \leq L^k |x_0 - x_*|.$$