True or false? (unless otherwise specified.) Questions 1–15: 1 point for a correct answer, 0 otherwise.

Answer grid

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
True	☒	☐	☐	☒	☒	☒	☐	☐	☒	☐	☒	☐	☐	☒	☒
False	☐	☒	☒	☐	☐	☐	☒	☒	☐	☒	☐	☒	☒	☐	☐

1. **Direct methods.** The computational cost of the backward substitution algorithm, for an upper triangular matrix of size $n \times n$, scales as $\mathcal{O}(n^2)$.

   True. Computing the i-th unknown requires $\mathcal{O}(n-i+1)$ operations, so the total cost scales as $\mathcal{O}(n^2)$.

2. **Direct methods.** To solve a linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, with A a general invertible matrix, computing first the inverse A^{-1} and then the product $\mathsf{A}^{-1}\boldsymbol{b}$ requires fewer floating point operations than computing the LU decomposition of A followed by forward and backward substitution.

   False. Both approaches have a cost of order n^3, but computing the inverse has a larger prefactor. It is also less numerically stable.

3. **Direct methods.** Once the LU decomposition $\mathsf{A} = \mathsf{LU}$ has been calculated, solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ for a new right-hand side $\boldsymbol{b}$ requires $\mathcal{O}(n^3)$ floating point operations.

   False. It suffices to solve $\mathsf{L}\boldsymbol{y} = \boldsymbol{b}$ by forward substitution and then $\mathsf{U}\boldsymbol{x} = \boldsymbol{y}$ by backward substitution, which requires only $\mathcal{O}(n^2)$ floating point operations. This is why the LU decomposition can be reused for several right-hand sides.

4. **Direct methods.** For a general matrix A of size $n \times n$, the computational cost of calculating the LU decomposition scales as $\mathcal{O}(n^3)$.

   True. The computational cost is $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ floating point operations.

5. **Direct methods.** If A is symmetric and positive definite, then there exists a lower triangular matrix C with positive diagonal entries such that $\mathsf{A} = \mathsf{C}\mathsf{C}^T$.

   True. This is the Cholesky decomposition.

6. **Conditioning.** For every invertible matrix A and every real number $\alpha \neq 0$, it holds that $\kappa_2(\alpha\mathsf{A}) = \kappa_2(\mathsf{A})$.

   True. We have $\kappa_2(\alpha\mathsf{A}) = \|\alpha\mathsf{A}\|_2\|\alpha^{-1}\mathsf{A}^{-1}\|_2 = |\alpha||\alpha|^{-1}\|\mathsf{A}\|_2\|\mathsf{A}^{-1}\|_2 = \kappa_2(\mathsf{A})$.

7. **Conditioning.** If the determinant of an invertible matrix A is very small, for example $|\det(\mathsf{A})| \leqslant 10^{-10}$, then its condition number $\kappa_2(\mathsf{A})$ is very large.

   False. For example, $\mathsf{A} = 10^{-5}\mathsf{I} \in \mathbf{R}^{2\times 2}$ satisfies $\det(\mathsf{A}) = 10^{-10}$, but $\kappa_2(\mathsf{A}) = \kappa_2(\mathsf{I}) = 1$ by the previous item.

8. **Conditioning.** The only matrix A such that $\kappa_2(\mathsf{A}) = 1$ is the identity matrix.

   False. For example, $\kappa_2(2\mathsf{I}) = \kappa_2(\mathsf{I}) = 1$.

9. **Spectral radius.** For every square matrix A, it holds that $\rho(\mathsf{A}) \leqslant \|\mathsf{A}\|_2$.

   True. If $\mathsf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\boldsymbol{v} \neq \boldsymbol{0}$, then $|\lambda|\|\boldsymbol{v}\|_2 = \|\mathsf{A}\boldsymbol{v}\|_2 \leqslant \|\mathsf{A}\|_2\|\boldsymbol{v}\|_2$.

10. **Spectral radius.** The spectral radius ρ is a norm on the space of $n \times n$ matrices.

    False. The nonzero matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has spectral radius 0.

11. **Richardson's method.** Suppose that A is symmetric and positive definite, and let $\boldsymbol{b} \in \mathbf{R}^n$. Consider Richardson's iteration for solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega\left(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)}\right). \tag{1}$$

    If $\omega \neq 0$ and the sequence $(\boldsymbol{x}^{(k)})$ converges to some vector $\boldsymbol{x}_\infty$, then $\mathsf{A}\boldsymbol{x}_\infty = \boldsymbol{b}$.

    True. Taking the limit $k \to \infty$ in (1) gives $\boldsymbol{x}_\infty = \boldsymbol{x}_\infty + \omega(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}_\infty)$, so $\omega(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}_\infty) = \boldsymbol{0}$.

12. **Richardson's method.** If A is symmetric positive definite and $\omega < 0$, then the iteration (1) converges to the solution of the linear system for every initial guess.

    False. For every eigenvalue $\lambda > 0$ of A, we have $1 - \omega\lambda > 1$, so $\rho(\mathsf{I} - \omega\mathsf{A}) > 1$.

13. **Richardson's method.** If A is symmetric with eigenvalues 1, 4 and 9, then the iteration (1) with $\omega = \frac{1}{4}$ converges to the solution of the linear system for every initial guess.

    False. The iteration converges for every initial guess if and only if $\rho(\mathsf{I} - \omega\mathsf{A}) < 1$. Here $\rho(\mathsf{I} - \frac{1}{4}\mathsf{A}) = \max\{\frac{3}{4}, 0, |1 - \frac{9}{4}|\} = \frac{5}{4} > 1$. Convergence requires $0 < \omega < \frac{2}{9}$.

14. **Richardson's method.** If A is symmetric positive definite, then the iteration (1) converges to the solution of the linear system for every initial guess if and only if the spectral radius satisfies $\rho(\mathsf{I} - \omega\mathsf{A}) < 1$.

    True. The error $\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}_*$ satisfies $\boldsymbol{e}^{(k)} = (\mathsf{I} - \omega\mathsf{A})^k \boldsymbol{e}^{(0)}$, and since $\mathsf{I} - \omega\mathsf{A}$ is symmetric, $(\mathsf{I} - \omega\mathsf{A})^k \to \mathsf{0}$ if and only if $\rho(\mathsf{I} - \omega\mathsf{A}) < 1$.

15. **Link with optimization.** If A is symmetric positive definite, then a vector $\boldsymbol{x}_*$ is the solution of the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{x}_*$ is the minimizer of the function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}.$$

    True. Since A is symmetric, $\nabla f(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x} - \boldsymbol{b}$, so $\mathsf{A}\boldsymbol{x}_* = \boldsymbol{b}$ if and only if $\nabla f(\boldsymbol{x}_*) = \mathbf{0}$. Since A is positive definite, the function f is strictly convex, so this critical point is its unique minimizer.

16. **Bonus (Julia).** Write (on paper) a Julia function `forward(L, b)` which solves the linear system $\mathsf{L}\boldsymbol{x} = \boldsymbol{b}$ by forward substitution, for an invertible lower triangular matrix `L` and a vector `b`.

    For example:

```julia
function forward(L, b)
    n = length(b)
    x = zeros(n)
    for i in 1:n
        x[i] = b[i]
        for j in 1:i-1
            x[i] -= L[i, j] * x[j]
        end
        x[i] /= L[i, i]
    end
    return x
end
```