True or false? (unless otherwise specified.)

Answer grid

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
True	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
False	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

1. **Direct methods.** The computational cost of the backward substitution algorithm, for an upper triangular matrix of size $n \times n$, scales as $\mathcal{O}(n^2)$.

2. **Direct methods.** To solve a linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, with A a general invertible matrix, computing first the inverse A^{-1} and then the product $\mathsf{A}^{-1}\boldsymbol{b}$ requires fewer floating point operations than computing the LU decomposition of A followed by forward and backward substitution.

3. **Direct methods.** Once the LU decomposition $\mathsf{A} = \mathsf{LU}$ has been calculated, solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ for a new right-hand side $\boldsymbol{b}$ requires $\mathcal{O}(n^3)$ floating point operations.

4. **Direct methods.** For a general matrix A of size $n \times n$, the computational cost of calculating the LU decomposition scales as $\mathcal{O}(n^3)$.

5. **Direct methods.** If A is symmetric and positive definite, then there exists a lower triangular matrix C with positive diagonal entries such that $\mathsf{A} = \mathsf{C}\mathsf{C}^T$.

6. **Conditioning.** For every invertible matrix A and every real number $\alpha \neq 0$, it holds that $\kappa_2(\alpha\mathsf{A}) = \kappa_2(\mathsf{A})$.

7. **Conditioning.** If the determinant of an invertible matrix A is very small, for example $|\det(\mathsf{A})| \leqslant 10^{-10}$, then its condition number $\kappa_2(\mathsf{A})$ is very large.

8. **Conditioning.** The only matrix A such that $\kappa_2(\mathsf{A}) = 1$ is the identity matrix.

9. **Spectral radius.** For every square matrix A, it holds that $\rho(\mathsf{A}) \leqslant \|\mathsf{A}\|_2$.

10. **Spectral radius.** The spectral radius ρ is a norm on the space of $n \times n$ matrices.

11. **Richardson's method.** Suppose that A is symmetric and positive definite, and let $\boldsymbol{b} \in \mathbf{R}^n$. Consider Richardson's iteration for solving the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega\left(\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^{(k)}\right). \tag{1}$$

    If $\omega \neq 0$ and the sequence $(\boldsymbol{x}^{(k)})$ converges to some vector $\boldsymbol{x}_\infty$, then $\mathsf{A}\boldsymbol{x}_\infty = \boldsymbol{b}$.

12. **Richardson's method.** If A is symmetric positive definite and $\omega < 0$, then the iteration (1) converges to the solution of the linear system for every initial guess.

13. **Richardson's method.** If A is symmetric with eigenvalues 1, 4 and 9, then the iteration (1) with $\omega = \frac{1}{4}$ converges to the solution of the linear system for every initial guess.

14. **Richardson's method.** If A is symmetric positive definite, then the iteration (1) converges to the solution of the linear system for every initial guess if and only if the spectral radius satisfies $\rho(\mathsf{I} - \omega\mathsf{A}) < 1$.

15. **Link with optimization.** If A is symmetric positive definite, then a vector $\boldsymbol{x}_*$ is the solution of the linear system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ if and only if $\boldsymbol{x}_*$ is the minimizer of the function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathsf{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}.$$

16. **Bonus (Julia).** Write (on paper) a Julia function `forward(L, b)` which solves the linear system $\mathsf{L}\boldsymbol{x} = \boldsymbol{b}$ by forward substitution, for an invertible lower triangular matrix `L` and a vector `b`.